# How numbers help students solve physics problems

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# Abstract

Previous research has found that introductory physics students perform far better on numeric problems than on otherwise equivalent symbolic problems.<sup>1,2</sup> This paper describes a framework to explain these differences developed by analyzing interviews with introductory physics students as they worked on analogous numeric and symbolic problems. It was found that information about the physical situation, as well as the problem solving process are represented in subtly different ways in numeric problems compared to symbolic problems. In almost every respect the inclusion of numbers makes information more transparent throughout the problem solving process.

#### I. INTRODUCTION

In introductory physics symbolic algebra is one of the main representations used to understand physical reality. Unfortunately many students have difficulty solving problems using only symbols. Introductory physics students are often much more successful on numeric problems than on analogous symbolic problems. On a problem studied in an earlier investigation students performed nearly 50 percentage points better on a numeric question than the otherwise equivalent symbolic problem.<sup>1</sup> While such a large difference is rare, double digit differences in score are common.<sup>2</sup>

An analysis of student work showed that many of the errors were consistent with students using symbols in ways that were inconsistent with their definition. For example, using a symbol defined to be a property of one object as if it were a property of a different object. These difficulties indicate that many students struggle using and understanding the algebraic equations that are such a vital part of physics problem solving.

This paper presents a framework which attempts to explain these difficulties in terms of the different ways information is encoded by symbols in numeric and symbolic problems. From the perspective of this framework numbers help students solve physics problems because numbers make it easy to distinguish symbol states such as the difference between a known and an unknown, and also minimize the possibility of symbol association confusion. Without numbers, symbols in purely symbolic problems must encode a greater amount of information. In addition, the interpretation of a symbol may change as the solution progresses. For example, there is no commonly used notation to denote when a symbol has transformed from an unknown quantity into a known quantity.

The idea of multiple interpretations for an algebraic symbol is not new in the field of mathematics education research. As part of the Concepts in Secondary Mathematics and Science (CSMS) project studying the mathematical ability of 3000 British school children between the ages of 11 and 16, Kuchemann was able to identify six different interpretations that children applied to letters in algebraic problems.<sup>3</sup>

- 1. Letter Evaluated A letter is assigned a numeric value from the outset
- 2. Letter not used The children ignore the letter
- 3. Letter used as an object The letter is regarded as a shorthand for an object or as an

object in its own right. For example, 2a is interpreted as two apples, in contrast to the interpretation of 2a as 2 times the number of apples.

- 4. Letter used as a specific unknown The children regard a letter as a specific but unknown number, and can operate upon it directly
- 5. Letter used as a generalized number The letter is seen as representing, or at least as being able to take, several values rather than one
- 6. Letter used as a variable The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values.

While the first two categories were improvised methods, the latter four methods reflect legitimate ways that symbols are used in algebra. Each of those latter four interpretations can be valid depending on the context in which the symbol is introduced. Similarly in physics, a letter can refer to the unit of a quantity, a specific known or unknown quantity, or as a variable in a general equation.

Bills suggests that it is the implicit change in the meaning of symbols without explicit notational change that is the source of confusion for many students in algebra.<sup>4</sup> In algebra she identified four common transitions when such implicit changes take place: 1) variable to an unknown to be found, 2) placeholder in a form to an unknown to be found, 3) unknown to be taken as a given to an unknown to be found, and 4) unknown to be taken as given to a variable.

Students often have diffulty making the transition between arithmetic, which primarily deals with numeric computation, and algebra, which has a much greater reliance on symbolic representation. Kieran studied how prior experience with arithmetic computation skewed students' understanding of the equal sign in algebra.<sup>5–7</sup> She found that many students did not view the equal sign as an equivalence symbol, but rather as a "do something symbol". This interpretation was a generalization of a procedure for solving arithmetic problems, in which the equal sign is commonly associated with a prompt to perform a computation. When asked to solve problems they sometimes violated the equivalence relation expressed by the equal sign when performing their mathematical actions. For example, consider the following arithmetic problem.

In an existing forest 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contains 1063 threes. How many trees were there before the new trees were planted?<sup>5</sup>

A student might write 1063 + 217 = 1280 - 425 = 855 while solving the problem. The equal sign is used procedurally by this student as a prompt to evaluate the previous expression, but violates the definition of the equal sign because it is not used as a statement of equivalence between the preceding and following expressions. Even though the proper understanding of the equal sign was not important for the solution to this particular question, such a misunderstanding could potentially hinder the symbolic representation of relationships required for many algebraic solutions.

Many college students also demonstrate difficulties using and understanding algebraic symbols. Clement and colleagues studied undergraduate students' ability to mathematically represent relationships between quantities.<sup>8</sup> In these studies he found that many of these students had difficulties coordinating the different meanings of the symbols in the mathematical equations. He and his colleagues have used variations of what has become known as the "Students and Professors" problem.

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

First-year engineering students scored roughly 60% on this question, with the most common error being a reversal of the correct quantitative relationship (6S=P instead of S=6P). While it might first appear that the error was due to carelessness, research involving a variety of techniques demonstrated that many who made the reversal error demonstrated that they conceptually understood the relationship but did not understand how to correctly represent the relationship algebraically. Many students used the symbols S and P in a way similar to Kuchemann's third category in which the letter is used as shorthand for the object. For example, 6S was viewed as six students, rather than six times the number of students. More recently Cohen and Kanim tested student and professor like problems on introductory physics students and found similar results.<sup>9</sup>

Soloway, Lochhead and Clement showed that in the context of writing a computer program that students were much more likely to correctly represent the algebraic relationship.<sup>10</sup>

A class of 100 students was given a students and professors like problem and was split so that half were assigned to write a computer program and half asked to write the algebraic equation. The students asked to write the computer program performed 24 percentage points better than the group asked to write the algebraic equation. This finding is consistent with the idea that many students are more comfortable in the context of numeric computation than with symbolic representation.

Trigueros and Ursini performed a study in which they developed a mathematics questionnaire designed to probe first year undergraduates understanding of the different meanings of
symbols in algebra. The students in the study were 164 first-year undergraduates from a
variety of non-science majors who had failed a classification test meant to measure students'
preparedness to take calculus. Each question was created to measure at least one aspect
of their understanding of the symbol as an unknown, symbol as a general number, and the
symbol in a functional relationship (related variables). They found that the majority of the
students in their sample had very poor understanding of these three ways of understanding an algebraic symbol. The rate of success along each of the three categories was about
50%. They concluded that many first-year undergraduate students still primarily hold a
non-algebraic perspective on the meaning of symbols.

Even if students enrolled in introductory physics possess an algebraic perspective and can interpret the different meanings of algebraic symbols, they will find that problem solving in physics is not just the solving of algebra word problems with physical objects. Equations in physics contain a large amount of information, much of which is implicitly encoded and context dependent. This encoding is so specific to physics that even mathematicians may have difficulty interpreting the meaning of symbolic physics equations. Page 12,13 Redish asked both physicists and mathematicians the following question:

If 
$$A(x,y) = K(x^2 + y^2)$$
 with K being a constant, then what is  $A(r,\theta)$ ?

He reports that while physicists normally recognize the transformation from Cartesian to cylindrical coordinates and answer  $A(r,\theta) = Kr^2$ , mathematicians commonly view it as a symbol substitution and answer  $A(r,\theta) = K(r^2 + \theta^2)$ . For physicists the letter chosen carries information about the type of quantity it is meant to represent. For them x and y represent the positions along two perpendicular axes, but that is not necessarily the case for mathematicians. Another example of the difference in interpretation between mathemati-

cians and physicists is the use of the letter x. In physics the letter x usually represents a distance or position, and therefore cannot be used to represent an unknown as is the case in many mathematics courses.

Sherin has uncovered a context dependent vocabulary used by physicists to interpret symbolic equations. Where the same are interpreted to output conceptual narratives. In some cases the same equation structure can be interpreted in different ways. For example, the equation  $F_{net} = ma$  is most commonly understood as a causal relationship, whereas the equation N = mg is most commonly understood to represent a balancing of two forces even though both equations have the same mathematical structure: [] = [].

In most introductory physics courses the range of mathematical abilities of the students is large. Some students may possess only basic algebraic skills and may mainly rely on arithmetic computational strategies when solving problems. Others who have a strong understanding of algebra as it is taught in math classes may be confused by the ways symbols are used in physics. This paper argues that student difficulties with purely symbolic problems are to some extent a result of confusions related to how information is encoded by symbols in physics. The way information is represented as well as what information is represented is different in numeric problems compared to purely symbolic problems. Some of the difficulties students have may be a result of inappropriately generalizing methods that work when solving numeric problems but do not when solving symbolic problems.

# II. PROBLEM SOLVING INTERVIEW PROTOCOL

As an extension of earlier work analyzing numeric and symbolic problems on final exams in a large introductory physics course, speak-aloud problem solving interviews with thirteen introductory physics students was performed.<sup>16</sup> The goal of these interviews was to uncover mechanisms that explain why numeric problems are often much easier than symbolic problems.

During the interviews each student was first given the symbolic version to solve while speaking aloud about their method to reach the solution. Whether correct or incorrect, the subject was asked questions to gauge their understanding of the symbols in the problem. If the subject had difficulties with the symbolic version, they were asked to solve the numeric version of the same question. If the subject was then able to solve the numeric version the subject was then asked to use their numeric solution to find the correct symbolic expression. The students were never told whether they found the correct or incorrect result. The same procedure was used for all of the questions.

The choice to give the symbolic version before the numeric version was informed by the earlier results for similar problems in the earlier study involving large student populations.<sup>1</sup> In that earlier work the numeric versions had an average score between 85-95%, while the symbolic versions had an average score between 45-60%. The expectation was that individual students would be unable to solve the symbolic version, but then later would be able to solve the numeric version of that same problem.

The analysis of the interviews focused on instances when errors were made and on instances when students were incorrect on one version, but correct on the other version. A framework describing differences in how information was encoded in numeric and symbolic problems was developed to describe the errors observed during the interviews. A more extensive discussion of the interviews can be found in the author's dissertation.<sup>15</sup>

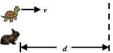
### A. Subjects

Thirteen subjects who were students in the calculus-based introductory mechanics course, Physics 211, at the University of Illinois were interviewed in spring 2007. The subjects were chosen based on their score on the  $1^{st}$  exam so that the interview sample reflected the range of abilities in the student population.<sup>16</sup> To ensure that kinematics was fresh in their minds, the interviews occurred no later than two weeks after the  $1^{st}$  exam. The students were paid \$15 and given two written solutions to past exams in Physics 211 for their participation.

#### B. Interview Questions

The physics questions used in this study were modified versions of the questions used in the earlier final exam study. The structure of the questions were the same but with different surface features. In the prior study, the performance by a large population of students was used to categorize errors, and so the likely errors were known in advance. The questions used during the interviews can be seen in Figures 1 and 2.

The tortoise and the hare have a race. The hare gets off to a very large lead, and when he gets near the finish line decides to take a nap. A few minutes later the tortoise moving at a constant speed of  $\nu$  passes the hare who is sleeping a distance d from the finish line. At the moment the tortoise passes, the hare wakes up and accelerates from rest with a constant acceleration to try to pass before the tortoise crosses the finish line.



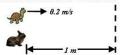
What is the minimum acceleration that the hare must have in order to catch the tortoise before it reaches the finish line?

A) 
$$a_{min} = (2v)^2/d$$
  
D)  $a_{min} = v^2/(2d)$ 

B) 
$$a_{min} = 2v^2/d$$
  
E)  $a_{min} = v^2/(4d)$ 

C) 
$$a_{min} = v^2/d$$

The tortoise and the hare have a race. The hare gets off to a very large lead, and when he gets near the finish line decides to take a nap. A few minutes later the tortoise moving at a constant speed of 0.2 m/s passes the hare who is sleeping a distance 1 m from the finish line. At the moment the tortoise passes, the hare wakes up and accelerates from rest with a constant acceleration to try to pass before the tortoise crosses the finish line.



What is the minimum acceleration that the hare must have in order to catch the tortoise before it reaches the finish line?

A) 
$$a_{min}$$
 = 0.16 m/s<sup>2</sup>  
D)  $a_{min}$  = 0.02 m/s<sup>2</sup>

B) 
$$a_{min} = 0.08 \text{ m/s}^2$$
  
E)  $a_{min} = 0.01 \text{ m/s}^2$ 

C) 
$$a_{min} = 0.04 \text{ m/s}^2$$

FIG. 1: The numeric and symbolic versions of the *Tortoise and Hare* question. This question is analogous to the *Bank robber* question from the 2006 study.<sup>1</sup>

These questions were designed to contain only simple equation structures to minimize manipulation errors. While it is likely that manipulating symbolic equations is a difficult for many introductory students, the intent of these interviews was to identify other mechanisms that explain why students have difficulty with symbolic problems.

### III. HOW SYMBOLS ENCODE MEANING

The errors students made, and the problem solving approaches students took were examined and a framework was developed to describe the different types of meaning that are encoded by symbols. Table I lists the different types of information encoded by symbols used in a typical introductory physics problem.

This framework distinguishes symbol characteristics as either static or dynamic during the problem solving process. The term *symbol properties* is used to denote static characteristics

A jet airliner on a runway can accelerate at a constant rate from 0 to a speed  $\mathbf{v}_1$  in t seconds. How far has the jet traveled,  $\mathbf{L}$ , when it reaches its minimum takeoff speed of  $(\mathbf{v}_1/2)$ ? (Assume constant acceleration.)

A) 
$$L = v_i t$$
 B)  $L = v_i t t 2$  C)  $L = v_i t t 4$  D)  $L = v_i t t 8$  E)  $L = v_i t t 16$ 

A jet airliner on a runway can accelerate at a constant rate from 0 to a speed 80 m/s in 90 s.. How far has the jet traveled, L, when it reaches its minimum takeoff speed of 40 m/s? (Assume constant acceleration.)

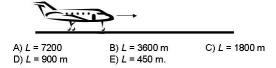


FIG. 2: The numeric and symbolic versions of the Airliner question. This question is analogous to the A car can qo question from the 2006 study.<sup>1</sup>

such as object, temporal, or spatial associations. And the term *symbol state* to denote whether the symbol is a variable, an unknown quantity, or a known quantity which can change as the solution progresses.

The symbol states category was informed by Kuchemann's categories of the meaning of algebraic letters discussed in the introduction. The scope of this category was limited to the interpretations commonly used in typical introductory physics problems. The symbol properties category was an expansion of Redish's description of how letter assignment is used to denote the type of quantity being represented.

The problem solving process typical in introductory physics can be described functionally as the assignment of properties to variables in general equations, and then the transformation of unknown symbols into known symbols. Symbols begin as variables in general equations, which rarely carry any specific subject/object, temporal, or axis associations. Once the variable is specified with definite associations it is either a known or unknown symbol. The goal of most traditional physics questions is the transformation of a target unknown into a known quantity. While students rarely confused symbol states and properties when solving the numeric problems, confusions were common when working on the symbolic problems.

The ease with which symbol states and properties could be interpreted during numeric and symbolic problems describe many of the difficulties observed during the student interviews,

Symbol States (Dynamic)	Example
1) General variable	1) A symbol in a general equation. $F$ in $F = ma$ .
2) Unknown	2) "Find the force, F"
3) Known	3) "A force of $F = 3N$ pushes"
Symbol Properties (Static)	Example
1) Type of Quantity	1) Mass, Force, Momentum, etc.
2) Object Association	2) Mass of the car, Mass of the bike.

3) Spatial and Temporal Associations 3) Initial, final, in, out.

TABLE I: Summary of the symbol states for a symbol used during physics problem solving, as well as the symbol properties that are used to specify the relationship of the symbol to the physical system.

which themselves were consistent with the errors observed in the larger population. The following sections describe mechanisms that were uncovered.

#### A. Symbol Association Errors

One general category of common errors was to confuse two different quantities of the same type. For example, using a symbol that is defined to represent car 1's velocity, as if it were car 2's velocity. Both quantities are velocities, but have different object associations. This type of confusion was common on symbolic problems because in purely symbolic solutions two different quantities of the same type can often be found in the same equation after equations are combined. However, in numeric sequential solutions, where each use of an equation results in a number that can be plugged into the next equation, this is impossible because only numeric quantities propagate from one equation to the next. Thus, the likelihood of such a confusion in numeric solutions is minimal.

This was a common error when solving the symbolic version of the Airliner question (shown in Figure 2). In this problem an airliner on a runway accelerates at a constant rate from rest. The airliner's final velocity and the time to reach that velocity are given, and the question asks you to find the distance traveled when the airliner reaches half the final

velocity.

There were many varieties of errors related to the confusion of two quantities of the same type. For example, even though the symbol v is defined as the final velocity, many students used it as if it were half the final velocity. Similarly the symbol t is defined as the time to reach the final velocity but was used as the time to reach half the final velocity. Two interview subjects made this latter error. These two students started by using the general equation  $x = x_i + v_i t + \frac{1}{2}at^2$ . They made the replacements  $x_i = 0$ ,  $v_i = 0$ , x = L, and a = v/t to get  $L = \frac{1}{2}(v/t)t^2$ . The students simplified this to the equation L = vt/2. Unfortunately the symbol t is used to represent two different times. The t from the general equation should properly represent the time to reach a speed v/2, and the t in the acceleration equation should represent the time the reach a velocity v. The students erred when they canceled the t's after combining the two equations.

Even though they made this error on the symbolic version, both were able to correctly solve the numeric version. The act of plugging in numbers seemed to act as a cue to specify the meaning of the variable they were replacing. Both attempted the same procedure as they had in the symbolic version, but as they were about to plug in a value for the time into the kinematics equation, they realized that it was necessary to solve for the specific time when the jet airliner reached a speed of 40 m/s (half the final speed), and that it was inappropriate to cancel the t's as they had in the symbolic version.

Subject F: Umm,  $x = \frac{1}{2}at^2$ , and then a = v/t ... so when I plugged in the x equals, which, a = v/t in my equation  $x = \frac{1}{2}at^2$ , I crossed out the times, but [the t in the acceleration equation] was for when it was 90 [m/s] and [the t in the general equation] is when, we don't know how long it took. So maybe I should...figure out... how long it takes for the plane to get to 40 m/s...[Subject F then correctly solves the numeric version]

Interviewer: OK so how confident do you feel about that?

Subject F: Umm, I was pretty confident, but I kind of got sidestepped over what time I should use, so I went to the side and solved for it.

These two students benefitted from the isolation of symbols with different properties afforded by the numeric procedure. In this example the inclusion of numbers allowed the students to isolate each meaning of the symbol t from the other definition by the use of

separate numeric equations.

### B. Variable confusion

The specification of symbol properties is the key step that differentiates a general equation with variables from an equation containing symbols with specific properties.

In purely symbolic problems the only way to distinguish a variable from a specific known or unknown quantity is with a subscript. Unfortunately, few students in the sample used them. Some students seemed to treat all symbols in the symbolic version as general variables.

While there is no special notation for variables in numeric problem solving, a general equation containing only general variables can be easily identified by the fact that it is purely symbolic, even though numbers are available. When numbers are plugged into the equation, the remaining symbols transform into specific unknown quantities. These symbols take on the symbol properties of the numeric quantities that are plugged into the equation. For example, if quantities for the car are plugged into the equation, then the remaining symbols must all represent unknowns with the properties of that car.

Students may inappropriately carry this interpretation of symbols from numeric problem solving to symbolic problem solving. There is evidence from the interviews some students believe that the symbols in purely symbolic equations represent general variables.

The failure to distinguish a variable from a specific unknown was found for some of the subjects to be the cause of the most common error of the Tortoise and the Hare problem (Figure 1). In this problem a Tortoise moving with a constant speed passes a Hare at rest a certain distance from the finish line. The instant the Tortoise passes, the Hare accelerates toward the finish line. Given the speed of the Tortoise and the distance from the finish line the question asks you to find the minimum acceleration needed for the Hare to catch up to the Tortoise before the finish line.

The most common error was to use the equation  $v_f^2 = v_i^2 + 2a\Delta x$ , to get the incorrect result  $a = v^2/(2L)$ . This is incorrect because the symbol v that is given as the velocity of the Tortoise is used as if it were the velocity of the Hare when it reaches the finish line. There were five students in the sample who made this error. Those five students were questioned in an attempt to determine if they had a consistent interpretation of the symbol. Two students said that the symbol represented the velocity of the Tortoise, and three said that

it represented the velocity of the Hare. The two who claimed that the symbol v was the velocity of the Tortoise later demonstrated a very dynamic interpretation of the symbol. In the following exchange, one of those two students switches her interpretation of v in less than a minute.

[Student selected the answer  $a_{min} = v^2/(2L)$ ]

**Interviewer:** OK and what does v represent? [points to the selected answer]

Subject A: umm, the veloc, the constant velocity of the tortoise

Interviewer: Alright, so let's say that instead of this question asking for the minimum acceleration, asked you to find ... um... the vel, the final velocity of the hare, do you think you could write down an equation for the final velocity of the hare when it reaches the finish line?

**Subject A:** umm, I think I would just, probably rearrange this equation [referring to her final answer of  $a_{min} = v^2/(2L)$ ]

Interviewer: OK

**Subject A:** Because it, blah, the acceleration does not change, I mean its constantly accelerating, but in this scenario its still the minimum acceleration, and distance doesn't change, so I would just rearrange the equation to find the final velocity.

Interviewer: And that would be the final velocity of the hare?

Subject A: Right

The two students who displayed this behavior seemed to treat the symbol associations as dynamic properties. They seemed to falsely equate the symbols in the final answer as general variables.

For the other three students who held a consistent yet incorrect definition of the symbol v, it is possible that the first time they considered the meaning of the symbol v was when they were asked about it. Because there are no specific cues (like plugging in numbers) in symbolic problems it is very easy to carry along symbols without considering their meaning.

### C. Confusion of known and unknown symbols

Many students also exhibited difficulties distinguishing known and unknown symbols when working on the symbolic versions. Unlike in mathematics courses where the symbol x in an equation is reserved for unknown quantities, in physics that symbol is already reserved for positions and distances, which can be variable, known, or unknown. During the interviews students often lost track of the known and unknown quantities. In some cases those confusions led students to inefficient and ineffective methods. For example, while solving the Tortoise and Hare symbolic problem three students started with the equations  $d = v_0 t$  and  $d = \frac{1}{2}at^2$ , and incorrectly eliminated the known quantity (d) leaving the two unknowns (a and t). As a result they ended up with one equation and two unknowns. All three were surprised when their final result was not one of the answer options.

**Subject I:** [Starts with equations  $d = v_0 t$  and  $d = \frac{1}{2}at^2$ ] um now solving for a, I actually solved for, so  $v_0 t = \frac{1}{2}at^2$ , let's start over, vt,  $2vt = at^2$  divided  $t^2$  [on one side of the equation], divided by  $t^2$  [on the other side of the equation], equals a, cross those guys out,  $2v_0/t = a$ , and that would give you none of the answers given, which stinks!... [Answer contains the unknowns a and b because he eliminated the known quantity d]

These types of errors were not observed in the numeric versions because known and unknown quantities were easily identified by either using a number or a letter, respectively. When solving a symbolic problem the distinction is implicit because both known and unknown symbols are represented by letters. Symbolic questions require that the solver actively identify the known quantities from the context in which the symbols are introduced.

#### IV. CONNECTION TO EARLIER STUDIES

The mechanisms described in the previous sections are consistent with the results from the earlier study comparing numeric and symbolic problems in a large population introductory physics course.<sup>2</sup> In that earlier work numeric and symbolic question pairs were analyzed to determine the question properties that were common for problems with large differences between the versions. The following question properties were common for questions where the symbolic version was significantly more difficult than the numeric version.

- Multiple equations This property distinguishes whether the problem is commonly solved with one equation or with multiple equations.
- General equation manipulation This property signifies whether it is possible to obtain one of the incorrect choices by combining general equations or manipulating a single general equation with minimal changes (for example, replacing x by d).
- Use of a compound expression This property signifies that to reach the correct symbolic solution students must replace a variable in a general equation with a compound expression (for example, replacing the variable v by a more specific compound expression v/2).
- Manipulation error This property signifies that a common error on this question is related to an incorrect manipulation of a symbolic equation.

The following sections describe the connection between the types of errors identified in the interviews and the results from the prior large population study.

#### A. Allowing for symbol association errors

Problems whose solution require multiple equations allows for the possibility of two or more symbols of the same type (two velocities for example) to be present in the same equation when equations are combined. This does not occur in numeric problems because only numbers propagate from one equation to the next. If a student solving a purely symbolic problem does not distinguish symbols of the same type with different associations, the student will be susceptible to making an error for problems involving multiple equations. On the other hand, problems involving only a single equation avoid the possibility of such a confusion.

Students who primarily solve numeric problems may not accustomed to paying attention to the possibility of such errors.

# B. Symbols treated as general variables

In the interviews some students used symbols as if all symbols were general variables with no specific associations. That observed error is consistent with the properties: general

A block of mass M on a frictionless horizontal surface is initially at rest. A constant horizontal force is applied to the block for T seconds, and the block reaches a final momentum P. How far did the block travel during this time period?

A) d = PT/(4m) B) d = PT/(2m) C) d = PT/m D) d = 2PT/m E) d = 4PT/m

A block of mass 8 kg on a frictionless horizontal surface is initially at rest. A constant horizontal force is applied to the block for 24 seconds, and the block reaches a final momentum 18 kg $^{\dagger}$ m/s. How far did the block travel during this time period?

A) d = 13.5 m. B) d = 27 m. C) d = 54 m. D) d = 108 m. E) d = 216 m.

FIG. 3: The numeric and symbolic versions of a question studied to understand how format influences strategic decisions.

equation manipulation, and use of a compound expression. It is plausible that if students view symbols as variables that any manipulation of a general equation containing the correct types of symbols would be a reasonable solution. Similarly, from this perspective the symbol v as a variable could be viewed as a more general version of the term v/2.

# C. Poor strategies resulting from symbol state confusion

While there were no question properties identified in the large population studies that directly related to the difficulty identifying unknowns, it seems plausible that such a confusion could prevent a student from correctly solving a symbolic question. One possible mechanism for an incorrect result would be that this confusion makes it more difficult to identify the correct strategy to reach the solution.

An analysis of strategies used to reach the correct answer was undertaken for the two versions of a problem (See Figure 3) administered in the prior large population study. The numeric and symbolic versions were given on two versions of the final exam of a calculus-based introductory physics course. A total of 765 students saw one of the two versions of this question. The numeric version had a average score of 79.6%. The symbolic version had an average score of 63.4%. The 16.2% difference between the versions is significant at the p < 0.001 level.

The final exam booklets were collected after the exam, and the students' written work was later analyzed. A difference in the types of strategies used to reach the correct answer

Numeric (N=58)	Symbolic (N=46)	Solution Strategy
69%	28%	Find velocity with $v = P/m$
17%	22%	Find the force with $F = P/t$
2%	15%	Start with $a = F/m$
0%	17%	Start with $a = v/t$
5%	11%	Dont know
3%	0%	No Work
3%	7%	Other

TABLE II: The breakdown of strategies used to find the correct answer on the numeric and symbolic versions. The sample analyzed were 58 on the numeric version and 46 on the symbolic version. The term "Find" means that they were able to solve for an unknown in terms of given quantities. The term "Start with" means they began with an equation that contained multiple unknowns.

were observed (See Table II). To study the differences in strategy this analysis focused only on those students who were able to find the correct final answer.

On the numeric version 69% of students with the correct solution used the strategy that began by solving for the velocity in terms of given quantities, v = P/m. The students then solved for the acceleration using the given time, a = P/(m \* t). And finally solved for the distance using  $d = \frac{1}{2}at^2$  to find that d = Pt/(2m). On the symbolic version this strategy was seen in only 28% of the correct solutions surveyed. In general, solutions that began with finding an unknown in terms of known quantities was found in 86% of the numeric solutions compared to in only 50% of the symbolic solutions. This result is consistent with the ease of identifying known quantities.

On the other hand, strategies that began with equations with multiple unknowns was seen in 32% of the symbolic solutions, compared to only 2% of numeric solutions. These strategies are not very efficient because these strategies contain at least one step where one set of unknowns is replaced with another set of unknowns. For example, on the symbolic version 15% of the solutions started with the equation a = F/m, which contain the unknowns a, and F. The students using this strategy then plugged this into an equation such as  $d = \frac{1}{2}at^2$ , to get the equation  $d = \frac{1}{2}(F/m)t^2$ , which contains the unknowns d and F. They then used the equation F = P/t to find d = Pt/(2m). Even though this leads to the correct result,

the replacement of one pair of unknowns for another pair of unknowns is a lateral step that leads no closer to the solution. Similar strategies starting with the equation a = v/t were also found.

It is plausible that if difficulties identifying known and unknown quantities in a problem can lead to inefficient strategies that lead to the correct result, then it can also lead to alternative strategies that do not.

#### V. DISCUSSION

The use of symbols in physics is complex and subtle. Physics instructors are comfortable using symbols in many different ways, as well as seamlessly transitioning between interpretations. Unfortunately, many students are not familiar with how symbol meaning changes, and can be confused by their use. As instructors, we need to be aware of how we use symbols, and also that because of our expertise such transitions between uses can be effortless. For example, the transition from a variable in a general equation to a quantity with specific associations can occur without a change in notation. Such transitions should be made explicit to introductory students.

There are also other transitions in symbolic meaning that are common and which often occur without commentary. One example is the derivation of a general result from a specific case. In the solution of the specific case, the symbols contain specific associations and are treated as known and unknown quantities. But in the process of generalizing the result, the symbols in the final equation are transformed into variables with no specific object associations.

Another example is the use of limiting cases to evaluate a symbolic result. Again, the solution of the problem involves symbols with specific associations treated as either known or unknown symbols. When considering limiting behavior the symbols retain their associations, but have variable value. This combination of symbol properties and symbol states is not seen during the earlier problem solving process.

The transition in the procedures from numeric to symbolic problem solving is also very subtle. While every step in a symbolic solution can be mapped to the numeric solution, the way information is represented as well as what information is represented changes from one format to the other. Students who are comfortable with numeric problem solving may

find the procedures they are familiar with do not apply to symbolic problems. For example, treating the symbols in purely symbolic questions as general variables may work in numeric solutions, but does not in symbolic solutions. And while tracking the associations of quantities is not important when solving numeric problems, it is very important when solving symbolic problems.

A common practice among physics instructors is to assign numeric problems, but to instruct the students to solve the problems symbolically and then only plug in the numbers as one of the last steps. While this is a sensible balance between numeric and symbolic problem solving, one should not assume that the students will be able to easily solve the problem symbolically without instruction or modeling of how to do so.

One of the main differences between numeric and symbolic problem solving is the necessity to keep track of symbol properties/associations. Luckily, this can be done easily with subscripts which distinguish symbols with specific properties and associations from just variables. In addition, the consideration of subscripts may act as a cue for students to consider the meaning of the symbols they are using, just as plugging in numbers did for the interview subjects. Unfortunately, it has been my experience that few students us them without explicit directions to do so.

Another difference is the extra care one must take to identify known and unknown symbols in symbolic problems. Although students may have implicitly learned from numeric problem solving that identifying known and unknown quantities is pointless because the ease of identification in numeric problems, it is much more helpful when solving symbolic problems. Some students may benefit from an explicit notation to distinguish unknowns, such as circling or underlining the unknown symbols.

For physics experts symbolic representations are very important. Symbolic solutions allow one to find a general solution, to identify important variables, and is a method that allows one to check for errors and sensibility using limiting cases. The importance for experts makes students' difficulties with symbolic problems all the more troublesome. If we expect students to continue on from introductory physics to a physics major, then it is important that they become comfortable using symbols.

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